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# Vacuum Expectation Values of the Energy-Momentum Tensor in Two Dimensions

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A general form of the vacuum expectation values of the components of the energy-momentum tensor is derived in the case of static two-dimensional space-time. Conditions for the regularity of the energy-momentum tensor are set. A generalized formula for the black hole temperature originally found by Hawking is given. The regularity of the energy-momentum tensor in the presence of more than one horizon is investigated.

## 1. INTRODUCTION

The renormalization of the energy-momentum tensor in curved space-time has recently been investigated from many points of view (Birrell and Davies, 1982, Chapter 6). This research has been stimulated by the remarkable discovery by Hawking (1975) of black hole evaporation through quantum effects. Even in the case of a Schwarzschild black hole the exact form of the renormalised energy-momentum tensor is not known [Hawking's calculation gives essentially the asymptotic behavior of this tensor at large values of the retarded time (DeWitt, 1975)].

However, the Hawking effect and in particular the characteristic black-body spectrum of emitted particles occurs in the two-dimensional model of a Schwarzschild black hole and in this case the form of the energy-momentum tensor can be given exactly (Christensen and Fulling, 1977). The aim of this paper is to derive the form of vacuum expectation values of the energy-momentum tensor for the general case of two-dimensional static space-time and investigate its properties. These space-times have as

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special cases two-dimensional specializations of most of the known black hole metrics. In Section 2 we examine the general form of the energy-momentum tensor and we give conditions ensuring its regularity on the null Killing vector orbits (horizons). In Section 3 we derive a generalized formula for the black hole temperature and we investigate the regularity of the energy-momentum tensor when there is more than one horizon present in space-time.

## 2. CONDITIONS FOR THE REGULARITY OF THE ENERGY-MOMENTUM TENSOR

We consider the following two-dimensional metric:

$$ds^2 = F(r)\, dt^2 - F^{-1}(r)\, dr^2 \tag{1}$$

This is the metric of a two-dimensional, totally geodesic, timelike surface in a static four-dimensional space-time. Such surfaces occur in Schwarzschild, Reissner–Nordström, and de Sitter space-time. The surface $r = r_H$ such that $F(r_H) = 0$ is the null Killing vector orbit and it is often called a horizon. We assume that the zeros of the function $F(r)$ are single and given by $r = r_i$, $i = 1, 2, \ldots, n < \infty$, with $0 < r_1 < r_2 < \cdots < r_n < \infty$. Each region $0 < r < r_1$, $r_1 < r < r_2, \ldots, r_n < r < \infty$ we shall call a block. The region such that $r \in (r_i, r_{i+1})$ we shall denote $B_i$.

We attempt to find the form of the vacuum expectation values of the renormalized energy-momentum tensor $\langle T_{\mu\nu} \rangle$ for massless fields in the space-time given by metric (1). We assume that $\langle T_{\mu\nu} \rangle$ is covariantly conserved, i.e.,

$$\langle T^{\mu\nu} \rangle_{;\mu} = 0 \tag{2}$$

and that $\langle T_{\mu\nu} \rangle$ has the anomalous trace given by (Birrell and Davies, 1982, Chapter 6)

$$\langle T^{\mu}_{\mu} \rangle = -\alpha R \tag{3}$$

where $R$ is the Ricci scalar and $\alpha$ is a constant dependent on the spin of the massless field ($\alpha = 1/24\pi$ for the scalar field).

In a block $B_i$ we introduce a generalized tortoise coordinate $r^*$:

$$r^* = \int^r \frac{dr}{F(r)} \tag{4}$$

In coordinates $(t, r^*)$ the metric (1) has the form

$$ds^2 = F(r)(dt^2 - dr^{*2}) \tag{5}$$

Integrating equation (2) and using equation (3), we have

$$\langle T_{\mu\nu}\rangle = \begin{bmatrix} T_{tt} & T_{tr^*} \\ T_{r^*t} & T_{r^*r^*} \end{bmatrix} = \begin{bmatrix} \langle T^\mu_\mu\rangle F(r) - H(r) & 0 \\ 0 & -H(r) \end{bmatrix} + A\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + B\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{6}$$

where

$$H(r) = \frac{1}{2}\int^r \langle T^\mu_\mu\rangle \frac{dF(r')}{dr'}\, dr'$$

(without a constant of integration) and $A$ and $B$ are constants of integration. Since $R = -F''$ (prime denotes differentiation with respect to $r$), we have $H = \frac{1}{4}\alpha F'^2$. The metric (1) is singular on the horizon $r = r_i$. However, this is only a coordinate singularity, since the curvature invariant $R$ is regular there. A general procedure to find a coordinate system regular on the event horizon $r = r_i$ was given by Walker (1970). One introduces Kruskal-like coordinates $U$ and $V$ which have the form

$$U = e^{-\kappa_i(t-r^*)}, \qquad V = e^{\kappa_i(t+r^*)} \tag{7}$$

where $\kappa_i = \frac{1}{2}F'(r_i)$ is a surface gravity on the horizon $r = r_i$.

The metric is given by

$$ds^2 = -\frac{g_i(r)}{\kappa_i^2}\exp[-2\kappa_i G_i(r)]\, dU\, dV \tag{8}$$

where $G_i(r)$ is a certain function regular on the horizon $r = r_i$ and $g_i(r) = F(r)/(r - r_i)$ [for example, for Schwarzschild space-time $\kappa_1 = (4M)^{-1}$, $g_1(r) = r^{-1}$, $G_1(r) = r$, and $r_1 = 2M$].

In $U$, $V$ coordinates the energy-momentum tensor (6) has the form

$$\langle T_{\mu\nu}\rangle = \begin{bmatrix} T_{UU} & T_{UV} \\ T_{VU} & T_{VV} \end{bmatrix} = g_i^{-2}(r_i)\begin{bmatrix} \dfrac{4A + F\langle T^\mu_\mu\rangle - 2H}{UU} & \dfrac{-F\langle T^\mu_\mu\rangle}{UV} \\ \dfrac{-F\langle T^\mu_\mu\rangle}{UV} & \dfrac{4B + F\langle T^\mu_\mu\rangle - 2H}{VV} \end{bmatrix} \tag{9}$$

Since the metric (1) is invariant under the discrete isometry $t \to -t$, the horizon $r = r_i$ consists of two segments defined by $t = +\infty$ and $t = \infty$. We shall denote the segment $r = r_i$, $t = +\infty$ by $H_i^+$ and the segment $r = r_i$, $t = -\infty$ by $H_i^-$. In coordinates $U$, $V$, $H_i^+$ is given by $U = 0$, $V$ finite when $F'(r_i) > 0$ and by $U$ finite, $V = 0$ when $F'(r_i) < 0$; $H_i^-$ is given by $U$ finite, $V = 0$ when $F'(r_i) > 0$ and by $U = 0$, $V$ finite when $F'(r_i) < 0$.

Our third assumption about the energy-momentum tensor is that $\langle T_{\mu\nu}\rangle$ must be regular on the horizon in the coordinate system regular there. Under the above assumption we have the following regularity conditions for components of $\langle T_{\mu\nu}\rangle$.

When $F'(r_i)>0$:

$$\text{on } H_i^+: \quad A=-[\tfrac{1}{4}F\langle T^\mu_\mu\rangle-\tfrac{1}{2}H]_{r=r_i}=\tfrac{1}{8}\alpha F'^2(r_i) \tag{10a}$$

$$\text{on } H_i^-: \quad B=\tfrac{1}{8}\alpha F'^2(r_i) \tag{10b}$$

When $F'(r_i)<0$:

$$\text{on } H_i^+: \quad B=\tfrac{1}{8}\alpha F'^2(r_i) \tag{11a}$$

$$\text{on } H_i^-: \quad A=\tfrac{1}{8}\alpha F'^2(r_i) \tag{11b}$$

The above conditions show that there does not exist a nontrivial, traceless energy-momentum tensor nonsingular on the horizon in the two-dimensional static space-time.

## 3. APPLICATIONS

In this section we apply the results of the preceding section to physical situations.

First we consider the case of a block for which $r\in(r_n,\infty)$ and assume that $F(\infty)=1$. This last assumption means that space-time is asymptotically flat. Such a block may be thought of as describing space-time outside matter. We can have three cases.

1. The matter does not undergo gravitational collapse and therefore there are no horizons $H^+$ and $H^-$. Thus the regularity conditions do not impose any restrictions on the values of constants $A$ and $B$ in (6). The case when constants $A$ and $B$ are set to zero corresponds to the situation when there is no incoming and outgoing radiation. In Schwarzschild space-time this corresponds to the Boulware vacuum $|0_S\rangle$. The energy-momentum tensor is interpreted as the vacuum polarization caused by the gravitational field.

2. The matter collapses to a black hole. There is only the future horizon $H^+$. In this case $A$ is fixed by (10a) and the constant $B$ is arbitrary. The case $B=0$ means that there is no incoming flux of particles on past null infinity $J^-$. At $J^+$, $\langle T^\mu_\nu\rangle$ has the form

$$\lim_{r\to\infty}\langle T^\mu_\nu\rangle=\tfrac{1}{8}\alpha F'^2(r_n)\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix} \tag{12}$$

This corresponds to the situation in which a black hole radiates a flux of massless particles of blackbody spectrum of temperature $T=(1/4\pi)F'(r_n)$.

In the case of the Schwarzschild black hole the chosen quantum state corresponds to the Unruh vacuum $|0_U\rangle$. The energy-momentum tensor found in this case agrees with the formula obtained by Hiscock (1980) by other methods.

3. We have no matter at all. There is an eternal black hole. Both horizons $H^+$ and $H^-$ exist. Thus both conditions (10a) and (10b) must be satisfied. At $r=\infty$, $\langle T^\mu_\nu\rangle$ has the form

$$\lim_{r\to\infty}\langle T^\mu_\nu\rangle=\tfrac{1}{8}\alpha F'^2(r_n)\begin{bmatrix}2 & 0\\ 0 & -2\end{bmatrix} \tag{13}$$

This corresponds to the situation in which the black hole is in an equilibrium with a gas of massless particles of temperature $T=(1/4\pi)F'(r_n)$. In the case of the Schwarzschild black hole the chosen quantum state corresponds to the Israel-Hartle-Hawking vacuum $|0_K\rangle$.

As a second application, let us consider the space-time that contains a block $B_i$ and horizons $H_i^+$, $H_i^-$, $H_{i+1}^+$, and $H_{i+1}^-$. In general the signs of $F'(r_i)$ and $F'(r_{i+1})$ are opposite. Without loss of generality we can suppose that $F'(r_i)<0$ and $F'(r_{i+1})<0$. Thus, if the energy-momentum tensor is to be regular on event horizons, the constants $A$ and $B$ must satisfy conditions (10a) and (10b) on horizons $H_{i+1}^+$ and $H_{i+1}^-$, respectively, and conditions (11b) and (11a) on horizons $H_i^-$ and $H_i^+$, respectively. Hence, we have the following conditions on the constants $A$ and $B$:

$$A=\tfrac{1}{8}\alpha F'^2(r_{i+1}),\qquad B=\tfrac{1}{8}\alpha F'^2(r_{i+1}) \tag{14}$$

$$A=\tfrac{1}{8}\alpha F'^2(r_i),\qquad B=\tfrac{1}{8}\alpha F'^2(r_i) \tag{15}$$

In general we have that $F'^2(r_i)\neq F'^2(r_{i+1})$. Thus, conditions (14) and (15) are incompatible. Thus, there does not exist an energy-momentum tensor regular on both $H_{i+1}^+$ and $H_i^-$ or on both $H_i^+$ and $H_{i+1}^-$. A special case of the above situation for the Reissner-Nordström space-time has been considered by Hiscock (1980) and he used his results in an argument in favor of the strong version of the cosmic censorship hypothesis.

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